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Throughflow effects in the Rayleigh–Bénard convective instability problem

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The effect of vertical throughflow on the onset of convection in a fluid layer, between permeable horizontal boundaries, when heated uniformly from below, is re-examined analytically. It is shown that when the Péclet number Q is large in magnitude, the critical Rayleigh number R_c is proportional to Q^n, where $n = 0$, 1, 2, 3 or 4, with a coefficient depending on the Prandtl number P, according to the types of boundaries. When the upper and lower boundaries are of different types, the effect of a small amount of throughflow in one direction is to decrease R_c. This is so when the throughflow is away from the more restrictive boundary. Contributions arise from the curvature of the basic temperature profile, and from the vertical transport of perturbation velocity and perturbation temperature. The decrease in R_c is small if $P \sim 1$ but can be of significant size if $P \ll 1$ or $P \gg 1$.

1. Introduction

The determination of the criterion for the onset of convection in a horizontal fluid layer heated uniformly from below is a classical problem associated with Lord Rayleigh and H. Bénard. The steady-state conduction solution becomes unstable, and convection begins, when the Rayleigh number R exceeds a certain critical value R_c. In the standard problem there is no flow of fluid across the horizontal boundaries. A modified problem, where the boundaries are permeable, and there is vertical throughflow produced by injection at one boundary and removal of fluid at the other boundary, was studied by Shvartsblat (1968, 1969, 1971) and his results were summarized by Gershuni & Zhukhovitskii (1976). As Shvartsblat pointed out, the problem is of interest because of the possibility of controlling the convective instability by adjustment of the transverse throughflow.

The throughflow is measured by a Péclet number Q. Shvartsblat found, for the case of conducting rigid permeable boundaries, that R_c was independent of the sign of Q, and increased markedly with Q increasing, i.e. the effect of throughflow is stabilizing and is independent of the direction of flow. Gershuni & Zhukhovitskii (1976, p. 236) wrote that the stabilizing effect may be explained as follows. With increasing injection velocity a temperature boundary layer forms at one of the boundaries. This decreases the effective thickness of the stratified layer of fluid which (at sufficiently large Q) is of order $d_{\text{eff}} \sim d/Q$, where d is the layer depth. On the other hand, the characteristic temperature difference across the layer remains fixed. The critical Rayleigh number, defined in terms of d, is thus of the order $R_c \sim (d/d_{\text{eff}})^3$, so that it increases with the Péclet number according to $R_c \sim Q^3$. Gershuni & Zhukhovitskii stated that the numerical results corroborate this estimate.

It is shown below that this argument is deceptive because it assumes that the perturbation temperature and velocity functions will retain the same shape, and this is not so in general because the effective boundary conditions are changed. Thus the asymptotic behaviour of R_c as Q becomes large depends on the nature of boundaries specified. The argument is also misleading in suggesting that the effect of throughflow is invariably stabilizing. When the upper and lower boundaries are of different types, a small amount of throughflow in one particular direction will be destabilizing. This is demonstrated and explained below.

The effect of throughflow is in general quite complex. Not only is the basic temperature profile altered, but in the perturbation equations contributions arise from the convection of both temperature and velocity, and there is an interaction between all of these contributions. The meteorologists Krishnamurti (1975) and Somerville & Gal-Chen (1979) have discussed the effects of small amounts of throughflow but their main interest in it was as a measure of a vertical asymmetry with which is associated the stability of hexagonal cells. The corresponding problem for convection in a saturated porous medium has been analysed by Wooding (1960), Sutton (1970), Homsy & Sherwood (1976) and Jones & Persichetti (1986). In the last paper the authors found that their numerical results indicated that in one situation a small amount of throughflow was destabilizing, but they were unable to explain this phenomenon. The present author's earlier note (Nield, 1987) was written to provide an explanation, for the porous-medium problem. The present paper reports an analysis for the viscous-fluid problem, in which the dependence on Prandtl number complicates the situation.

In general the determination of the critical Rayleigh number in a given situation entails solving the eigenvalue problem by numerical means, a process which is time consuming and which may produce less than full understanding. Fortunately there is one situation when the solution for arbitrary Q can be found in simple closed form. This is so when both boundaries are such that at them the heat flux is held constant, and so the perturbation heat flux is zero, i.e. the boundaries are 'insulating' with respect to temperature perturbations. This fact is exploited in the following analysis. For other types of boundary a perturbation approach is used to solve the problem for small values of Q.

2. Analysis

The relevant equations are those given by Gershuni & Zhukhovitskii (1976, p. 235). It is known that instability appears in non-oscillatory form. The non-dimensional equations for the amplitudes of the perturbation vertical velocity W and the perturbation temperature Θ can be written as

$$(\mathrm{D}^2-a^2)^2W-\frac{Q}{P}\mathrm{D}(\mathrm{D}^2-a^2)\,W = Ra^2\Theta, \tag{1}$$

$$(\mathrm{D}^2-a^2)\,\Theta-Q\mathrm{D}\Theta = FW. \tag{2}$$

Here $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}z$, where z is the upwards vertical coordinate and a is the horizontal wavenumber of the disturbance. The Prandtl number P, the Péclet number Q and the Rayleigh number R are defined by

$$P = \frac{\nu}{\kappa}, \quad Q = \frac{V_0\, d}{\kappa}, \quad R = \frac{g\alpha d^3\,\Delta T}{\nu\kappa}, \tag{3}$$

where ν is the kinematic viscosity, κ the thermal diffusivity, V_0 the imposed vertical throughflow velocity, d the layer depth, g the gravitational acceleration, α the volumetric expansion coefficient and ΔT the imposed temperature difference. The non-dimensional gradient F of the steady-state basic temperature distribution is given by

$$F = \frac{-Q\,e^{Qz}}{e^{Q}-1}. \tag{4}$$

The differential equations (1) and (2) must be solved subject to appropriate boundary conditions. As usual, we suppose that the boundaries, at $z = 0$ (lower) and $z = 1$ (upper), are either 'rigid' or 'free', and either 'conducting' or 'insulating'. At a rigid boundary $W = \mathrm{D}W = 0$, and at a free boundary $W = \mathrm{D}^2W = 0$. At a conducting boundary $\Theta = 0$, and at an insulating boundary $\mathrm{D}\Theta = 0$. For given boundary conditions, R is obtained as an eigenvalue.

2.1. *Solution for the case of insulating boundaries*

When both boundaries are insulating, it is easily checked that R attains its minimum value R_c as a varies when $a = 0$. Accordingly, an expansion in powers of a^2 is appropriate. Thus we let

$$(W, \Theta) = (W_0, \Theta_0) + a^2(W_1, \Theta_1) + \ldots$$

and substitute in (1) and (2) and in the boundary conditions.

The zero-order equations have solution

$$W_0 = 0, \quad \Theta_0 = 1.$$

The first-order equations are then

$$\mathrm{D}^4W_1 - \mu \mathrm{D}^3 W_1 = R, \tag{5}$$

$$\mathrm{D}^2\Theta_1 - Q\mathrm{D}\Theta_1 = 1 + FW_1, \tag{6}$$

where, for convenience, we have defined

$$\mu = \frac{Q}{P}. \tag{7}$$

The solvability requirement is that

$$\langle RW_0 \rangle = 0, \quad \langle (1+FW_1)\,\Theta_0 \rangle = 0,$$

where

$$\langle f \rangle \equiv \int_0^1 f\,\mathrm{d}z.$$

The first condition is satisfied trivially. The second requires that

$$\langle 1 + FW_1 \rangle = 0. \tag{8}$$

The general solution of (5) is

$$W_1 = c_0 + c_1 z + c_2 z^2 + c_3\,e^{\mu z} - \frac{Rz^3}{6\mu}. \tag{9}$$

For the case of two rigid boundaries,

$$W_1 = \mathrm{D}W_1 = 0 \quad \text{at} \quad z = 0, 1.$$

Hence

$$\{c_0, c_1, c_2, c_3\} = \tfrac{1}{6}R[2\mu+\mu^2-(2\mu-\mu^2)\,e^{\mu}]^{-1}\{-1, -\mu, 3+2\mu-(3-\mu)\,e^{\mu}, 1\},$$

For the case of two free boundaries,

$$W_1 = D^2W_1 = 0 \quad \text{at} \quad z = 0, 1.$$

Hence

$$\{c_0, c_1, c_2, c_3\} = \tfrac{1}{6}R[-\mu+\mu^3\,e^{\mu}]^{-1}\{-6, 6+2\mu^2-(6-\mu^2)\,e^{\mu}, -3\mu^2, 6\}.$$

If the lower boundary is rigid and the upper one free then $W_1 = DW_1 = 0$ at $z = 0$, and $W_1 = D^2W_1 = 0$ at $z = 1$. Hence

$$\{c_0, c_1, c_2, c_3\} = \tfrac{1}{6}R[2\mu+2\mu^2-(2\mu-\mu^3)\,e^{\mu}]^{-1}\{-4, -4\mu, 6+6\mu-(6-\mu^2)\,e^{\mu}, 4\}.$$

The expression for W_1 thus determined can now be substituted into (8) to yield a formula for R in terms of Q and μ. Finally μ can be replaced by Q/P. For the symmetrical cases, R is then an even function of Q, as one would expect. Thus one obtains the following results for the values of R, which in fact are those for R_c. The suffix c is dropped in the rest of the paper.

2.1.1. *Rigid–rigid*

$$R = 12Q^3\,\mathrm{sh}\,(\tfrac{1}{2}Q)\,[\mu^2\,\mathrm{ch}\,(\tfrac{1}{2}\mu)-2\mu\,\mathrm{sh}\,(\tfrac{1}{2}\mu)]\Big/\left\{\left[\frac{\mu^2Q^2}{Q+\mu}+6Q+6\mu\right]\mathrm{sh}\,(\tfrac{1}{2}(Q+\mu))\right.$$
$$\left.-[4\mu Q+12]\,\mathrm{ch}\,(\tfrac{1}{2}(Q+\mu))-[2\mu Q-12]\,\mathrm{ch}\,(\tfrac{1}{2}(Q-\mu))-6[Q-\mu]\,\mathrm{sh}\,(\tfrac{1}{2}(Q-\mu))\right\}. \quad (10)$$

Hence $$R \sim 6Q(Q+\mu) \quad \text{as} \quad Q \to \pm\infty.$$

Also $$R \sim 720\left(1+\frac{Q^2}{42}-\frac{\mu Q}{140}+\frac{\mu^2}{140}\right) \quad \text{as} \quad Q \to 0.$$

(The reader should recall that μ is an abbreviation for Q/P.) The quadratic form in the last expression is positive definite, and attains a minimum when $3\mu = 20Q$, that is $P = \frac{3}{20}$.

Hence in this case throughflow is always stabilizing, and the direction of flow does not matter.

2.1.2. *Free–free*

$$R = 24Q^3\,\mathrm{sh}\,(\tfrac{1}{2}Q)\,\mu^3\,\mathrm{sh}\,(\tfrac{1}{2}\mu)\Big/\left\{12\left[\frac{\mu^2Q^2}{Q+\mu}-Q-\mu\right]\mathrm{sh}\,(\tfrac{1}{2}(Q+\mu))\right.$$
$$\left.+[4\mu^2Q^2+12Q^2+12\mu^2]\,\mathrm{ch}\,(\tfrac{1}{2}(Q+\mu))+[2\mu^2Q^2-12Q^2-12\mu^2]\,\mathrm{ch}\,(\tfrac{1}{2}(Q-\mu))\right\}. \quad (11)$$

Hence $$R \sim 3\mu Q \quad \text{as} \quad Q \leftarrow \pm\infty.$$

Also $$R \sim 120\left(1+\frac{Q^2}{56}-\frac{\mu Q}{252}+\frac{\mu^2}{56}\right) \quad \text{as} \quad Q \to 0.$$

Again the quadratic form is positive definite. This time it has a minimum at $P = \frac{1}{9}$.

P	Q^*	$\frac{R-R_0}{R_0}$
0	−0	−0.102
0.01	−0.02	−0.101
0.1	−0.29	−0.092
0.2	−0.54	−0.079
0.5	−0.81	−0.034
1	−0.28	−0.002
1.25	0	0
10	1.10	−0.027
100	1.20	−0.033
∞	1.21	−0.034

TABLE 1. Values of Péclet number Q^* yielding the minimum critical Rayleigh number R as Q is varied, and of the corresponding proportional change in the critical Rayleigh number, for various values of the Prandtl number P, for the case of insulating boundaries, the lower being rigid and the upper free. R_0 (= 320)is the critical Rayleigh number for zero throughflow ($Q = 0$). The values have been calculated from the approximate formula (13).

2.1.3. *Rigid–free*

$$R = \frac{6Q^3(e^Q-1)\,[2\mu+2\mu^2-(2\mu-\mu^3)\,e^{\mu}]}{A\,e^{Q+\mu}+B\,e^Q+C\,e^{\mu}+D}, \tag{12}$$

where

$$\begin{aligned} A &= \frac{4Q^4}{Q+\mu}-4Q^3+6Q^2-12+\mu^2Q^2-4\mu^2Q+6\mu^2,\\ B &= -6Q^2+12-2\mu Q^2+12\mu,\\ C &= 12Q+12-2\mu^2Q-6\mu^2,\\ D &= \frac{-4Q^4}{Q+\mu}+4Q^3-12Q-12-4\mu Q^2-12\mu Q-12\mu. \end{aligned}$$

Hence

$$\begin{aligned} R &\sim 6\mu Q \quad \text{as} \quad Q\to\infty,\\ R &\sim 3Q(Q+\mu) \quad \text{as} \quad Q\to-\infty. \end{aligned}$$

Also

$$R \sim 320\left\{1-\frac{Q}{18}+\frac{5\mu}{72}+\frac{1040Q^2-431\mu Q+536\mu^2}{45360}\right\} \quad \text{as} \quad Q\to 0. \tag{13}$$

Again the quadratic expression is positive definite. The effect of a large throughflow is stabilizing, but the magnitude depends on whether Q is positive or negative. For small Q, upflow (Q positive) is stabilizing if $P < \frac{5}{4}$ and destabilizing if $P > \frac{5}{4}$, and the reverse is true for downflow. Comparison with calculations using the exact solution (12) shows that the approximate solution (13) is useful for determining the minimum value of R as Q varies for a given value of P. The values given in table 1 have an error of less than 10%. The table shows that very little destabilization is possible if the value of P is approximately one, but a reduction in critical Rayleigh number of 3% is possible if P is large, and a reduction of up to 10% is possible if P is small. These amounts arise from a difference in dynamic boundary condition. Further reduction is possible if the lower rigid boundary is conducting and the free upper boundary is insulating (see below).

Boundary conditions	R_0 exact	R_0 approx	Δ_b	Δ_m	Δ_t
RC, FC	1100.657	1138	$-0.038Q$	0.037μ	0
RI, FI	320	320	$-0.055Q$	0.070μ	0
RC, RI	1295.781	1452	$-0.050Q$	0	$-0.104Q$
RC, FI	669.001	692	$-0.104Q$	0.046μ	$-0.136Q$
RI, FC	816.748	953	$+0.012Q$	0.044μ	$+0.127Q$
FC, FI	384.693	413	$-0.068Q$	0	$-0.172Q$

Boundary conditions	a_0 exact	Δ'_m	Δ'_t
RC, FC	2.68	2.53μ	0
RI, FI	0	0	0
RC, RI	2.55	0	$-8.91Q$
RC, FI	2.09	0.82μ	$-4.43Q$
RI, FC	2.21	0.82μ	$+4.43Q$
FC, FI	1.76	0	$-2.05Q$

TABLE 2. Effect of small amounts of throughflow in cases of non-symmetric boundary conditions. RC, FI indicates that the lower boundary is rigid and conducting and the upper boundary is free and insulating. R_0 and a_0 are the critical Rayleigh number and the corresponding wavenumber when $Q = 0$. The 'exact' values of these are well known, and are those given in Platten & Legros (1984, p. 349). Δ_b, Δ_m and Δ_t are the proportional changes in R arising from the basic temperature profile, momentum transport and perturbation temperature transport respectively. Thus $R = R_0(1+\Delta_b+\Delta_m+\Delta_t)$. Similarly Δ'_m and Δ'_t are the proportional changes in a due to momentum transport and temperature transport respectively. Thus $a = a_0(1+\Delta'_m+\Delta'_t)$. The Δ values and the approximate value of R_0 have been obtained from (14) using the exact values of a_0 for the wavenumber.

2.2. *Approximate solutions for other non-symmetric boundary conditions*

When the boundaries are not insulating an exact closed-form solution for arbitrary values of Q is not obtainable, but for small values of Q an approximate solution can be obtained by a perturbation approach. The eigenvalue R for small non-zero Q can be found using the eigenfunction for zero Q. This approach can be combined with Galerkin approximation. It is known that the one-term Galerkin expansion used below, with an appropriate polynomial as trial function, yields for zero Q a value of R which in the worst case is too high by 16% (Platten & Legros 1984, table V1.2). (As a referee pointed out, it is also possible to proceed by an expansion of the form $R = R_0+\mu R_{10}+QR_{01}+\dots$, but the present approach provides sufficient accuracy and has the advantage that one can easily keep separate track of two effects involving Q.)

When one puts $W = AW_1$, $\Theta = B\Theta_1$, (where W_1 and Θ_1 are trial functions which satisfy the boundary conditions), substitutes into (1) and (2), multiplies (1) by W_1 and (2) by Θ_1, then integrates each term from $z = 0$ to 1, and performs some integration by parts, one obtains the following equation (when subscripts are dropped):

$$R = \frac{\{\langle(\mathrm{D}^2W)^2+2a^2(\mathrm{D}W)^2+a^4W^2\rangle+\mu\langle \mathrm{D}W(\mathrm{D}^2-a^2)\,W\rangle\}\,\{\langle(\mathrm{D}\Theta)^2+a^2\Theta^2\rangle-Q\langle\Theta \mathrm{D}\Theta\rangle\}}{a^2\langle W\Theta\rangle\,\langle -FW\Theta\rangle}. \tag{14}$$

It is immediately clear that three contributions from throughflow may be distinguished. The term with coefficient $\mu = Q/P$ arises from the vertical transport of momentum. The term with coefficient Q arises from the vertical transport of perturbation temperature. The term involving F arises from the curvature of the basic temperature gradient. In general, since the eigenfunction (W, Θ) depends on Q, there will be an interaction between the three contributions. However, when one takes a perturbation about the solution for zero Q, these contributions are isolated.

The following trial functions are excellent approximations, satisfying the boundary conditions exactly, for $Q = 0$:

for both boundaries rigid, $W = z^4 - 2z^3 + z^2$,

for both boundaries free, $W = z^4 - 2z^3 + z$,

for lower boundary rigid and upper boundary free, $W = z^4 - \frac{5}{2}z^3 + \frac{3}{2}z^2$,

for both boundaries conducting, $\Theta = z^2 - z$,

for both boundaries insulating, $\Theta = 1$,

for lower boundary conducting and upper boundary insulating, $\Theta = z^2 - 2z$.

If the boundaries are swapped, then z is replaced by $1 - z$.

The results of calculations from (14), using the appropriate selection of trial functions, are summarized in table 2.

3. Discussion

When we compare our results with the picture presented by Gershuni & Zhukhovitskii (1976) we see that their picture should be amended in two important respects.

First, their asymptotic relationship, $R \sim Q^3$, as Q becomes large, does not always hold. The term $\langle -FW\Theta \rangle$, which appears in the denominator of the expression for R in (14), can readily be expressed (using integration by parts) as an asymptotic power series in Q^{-1}. The coefficient of Q^{-m} is proportional to the m^{th} derivative of $W\Theta$ calculated at that boundary which is downstream to the throughflow. If that boundary is rigid and conducting then the coefficients of Q^0, Q^{-1} and Q^{-2} all vanish, and so $\langle -FW\Theta \rangle \sim Q^{-3}$. This is an exceptional case, the one for which three boundary conditions happen to make the first three coefficients zero. For a boundary that is not rigid, or one that is not perfectly conducting, $\langle -FW\Theta \rangle \sim Q^{-2}$, Q^{-1} or Q^0 according to whether the lowest-order non-zero boundary derivative of $W\Theta$ is the second, first or zeroth. For symmetrical boundaries, the numerator in (14) is of order Q^0. In other cases it is of order Q^0 or Q^1. Hence R is of order Q^n, where n may be 0, 1, 2, 3 or 4. For the case of two rigid boundaries, which is the case considered by the Russian authors, $R \sim Q^3$, in accordance with their computations. For two rigid insulating boundaries, or two free conducting ones, $R \sim Q^2$. For two free insulating boundaries, $R \sim Q$. One extreme situation, where $R \sim Q^4$, arises when the boundary downstream to the throughflow is rigid and conducting while the upstream boundary is rigid and insulating. The porous-medium problem, for the case of two porous insulating boundaries, provides an example where $R \sim Q^0$.

The second new feature is that when the boundaries are of different types, a small amount of throughflow can produce some destabilization. The results presented in table 2 show that this can be produced by three mechanisns. The curvature of the basic temperature profile produces destabilization when the throughflow is away from the more restrictive boundary. The effective temperature gradient is now confined to a shallower layer of fluid, but the stabilizing effect of this change is more than offset by a relaxation of an effective boundary restriction. (An alternative explanation is that the distortion of the basic temperature profile leads to large values of W where Θ is large, and hence to an increased rate of transfer of energy into the disturbance.)

The effects produced by the rigid–free (RF) and conducting–insulating (CI) contrasts are approximately additive; they almost double up in the RC, FI case and almost cancel in the RI, FC case. This effect from the basic temperature profile is accentuated by the contribution from the vertical transport of perturbation temperature. When the throughflow is from a conducting to an insulating boundary the effect of this transport is to oppose the stabilizing effect of thermal diffusion. Similarly the effect of throughflow from a rigid to a free boundary is to cause a transport of momentum which opposes the stabilizing effect of viscous diffusion. This last effect of throughflow is large when the Prandtl number is small.

When the Prandtl number P has a value close to one the amount of destabilization cannot be large, but this is not the case when P is either large or small. We have seen that a reduction of the critical Rayleigh number by 10% can result from a rigid–free differential, and this should be increased when there is also a conducting–insulating differential effect operating. It appears that experimental data are not available.

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